Necessary Conditions for Fredholmness of Singular Integral Operators with Shifts and Slowly Oscillating Data

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Dedicated to Vladimir S. Rabinovich on the occasion of his 70th birthday

Abstract. Suppose α is an orientation-preserving diffeomorphism (shift) of $\mathbb{R}_+ = (0,\infty)$ onto itself with the only fixed points 0 and ∞. In [6] we found sufficient conditions for the Fredholmness of the singular integral operator with shift

$$(aI - bW_\alpha)P_+ + (cI - dW_\alpha)P_-$$

acting on $L^p(\mathbb{R}_+)$ with $1 < p < \infty$, where $P_\pm = (I \pm S)/2$, S is the Cauchy singular integral operator, and $W_\alpha f = f \circ \alpha$ is the shift operator, under the assumptions that the coefficients a, b, c, d and the derivative α' of the shift are bounded and continuous on $\mathbb{R}_+$ and may admit discontinuities of slowly oscillating type at 0 and ∞. Now we prove that those conditions are also necessary.

Keywords. Orientation-preserving non-Carleman shift; Cauchy singular integral operator; slowly oscillating function; limit operator; Fredholmness.

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1. Introduction

A bounded linear operator on a Banach space is said to be Fredholm if its image is closed and the dimensions of its kernel and the kernel of its adjoint operator are finite. Let $\mathbb{R}_+ := (0, +\infty)$. A bounded and continuous function f on $\mathbb{R}$ is called slowly oscillating (at 0 and ∞) if for each (equivalently, for some) $\lambda \in (0, 1)$,

$$\lim_{r\to s} \sup_{t,\tau\in[\lambda r,r]} |f(t) - f(\tau)| = 0 \quad (s \in \{0, \infty\}).$$

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The set $SO(\mathbb{R}_+)$ of all slowly oscillating functions forms a C^*-algebra. This algebra properly contains $C(\overline{\mathbb{R}}_+)$, the C^*-algebra of all continuous functions on $\overline{\mathbb{R}}_+ := [0, +\infty]$. Suppose α is an orientation-preserving diffeomorphism of $\mathbb{R}_+$ onto itself, which has only two fixed points 0 and ∞. We say that α is a slowly oscillating shift if $\log \alpha'$ is bounded and $\alpha' \in SO(\mathbb{R}_+)$. The set of all slowly oscillating shifts is denoted by $SOS(\mathbb{R}_+)$.

Through the paper we suppose that $1 < p < \infty$ and $1/p + 1/q = 1$. It is easy to see that if $\alpha \in SOS(\mathbb{R}_+)$, then the shift operator W_α defined by $W_\alpha f = f \circ \alpha$ is bounded and invertible on all spaces $L^p(\mathbb{R}_+)$ and its inverse is given by $W_\alpha^{-1} = W_\beta$, where $\beta := \alpha_{-1}$ is the inverse function to α. It is well known that the Cauchy singular integral operator S given by

$$(Sf)(t) := \lim_{\varepsilon \to 0} \frac{1}{\pi i} \int_{\mathbb{R}_+ \setminus (t-\varepsilon, t+\varepsilon)} \frac{f(\tau)}{\tau - t} \, d\tau \quad (t \in \mathbb{R}_+)$$

is bounded on all Lebesgue spaces $L^p(\mathbb{R}_+)$ for $1 < p < \infty$. Put $P_\pm := (I \pm S)/2$.

By $M(\mathfrak{A})$ denote the maximal ideal space of a unital commutative Banach algebra $\mathfrak{A}$. Identifying the points $t \in \overline{\mathbb{R}}_+$ with the evaluation functionals $t(f) = f(t)$ for $f \in C(\overline{\mathbb{R}}_+)$, we get $M(C(\overline{\mathbb{R}}_+)) = \overline{\mathbb{R}}_+$. Consider the fibers

$$M_s(SO(\mathbb{R}_+)) := \left\{ \xi \in M(SO(\mathbb{R}_+)) : \xi|_{C(\overline{\mathbb{R}}_+)} = s \right\}$$

of the maximal ideal space $M(SO(\mathbb{R}_+))$ over the points $s \in \{0, \infty\}$. By [8, Proposition 2.1], the set

$$\Delta := M_0(SO(\mathbb{R}_+)) \cup M_\infty(SO(\mathbb{R}_+))$$

coincides with $\operatorname{clos}_{SO^*} \mathbb{R}_+ \setminus \mathbb{R}_+$ where $\operatorname{clos}_{SO^*} \mathbb{R}_+$ is the weak-star closure of $\mathbb{R}_+$ in the dual space of $SO(\mathbb{R}_+)$. Then $M(SO(\mathbb{R}_+)) = \Delta \cup \mathbb{R}_+$. In what follows we write $a(\xi) := \xi(a)$ for every $a \in SO(\mathbb{R}_+)$ and every $\xi \in \Delta$. This paper is a continuation of our work [6], where the following result was proved.

Theorem 1.1 ([6, Theorem 1.2])**.** *Let $a, b, c, d \in SO(\mathbb{R}_+)$ and $\alpha \in SOS(\mathbb{R}_+)$. The singular integral operator*

$$N := (aI - bW_\alpha)P_+ + (cI - dW_\alpha)P_- \tag{1.1}$$

with the shift α is Fredholm on the space $L^p(\mathbb{R}_+)$ if the following two conditions are fulfilled:

(i) *the functional operators $A_+ := aI - bW_\alpha$ and $A_- := cI - dW_\alpha$ are invertible on the space $L^p(\mathbb{R}_+)$;*

(ii) *for every pair $(\xi, x) \in \Delta \times \mathbb{R}$,*

$$\begin{aligned} n_\xi(x) := &\left[a(\xi) - b(\xi)e^{i\omega(\xi)(x+i/p)}\right] \frac{1 + \coth[\pi(x + i/p)]}{2} \\ &+ \left[c(\xi) - d(\xi)e^{i\omega(\xi)(x+i/p)}\right] \frac{1 - \coth[\pi(x + i/p)]}{2} \neq 0, \end{aligned} \tag{1.2}$$

where $\omega(t) := \log[\alpha(t)/t] \in SO(\mathbb{R}_+)$.

It turns out that the sufficient conditions for the Fredholmness of the operator N contained in Theorem 1.1 are also necessary.

Theorem 1.2 (Main result)**.** *Suppose $a, b, c, d \in SO(\mathbb{R}_+)$ and $\alpha \in SOS(\mathbb{R}_+)$. If the operator N given by* (1.1) *is Fredholm on $L^p(\mathbb{R}_+)$, then conditions* (i) *and* (ii) *of Theorem* 1.1 *are fulfilled.*

The proof of Theorem 1.2 is based on the method of limit operators, which was essentially developed by V. S. Rabinovich (see, e.g., [1, 12, 10] and the references therein), and on the Allan-Douglas localization (see [3]). The paper is organized as follows. In Section 2 we collect properties of slowly oscillating functions and slowly oscillating shifts. In Section 3 we recall properties of Mellin convolution operators with piecewise continuous and semi-almost periodic symbols. In Section 4 we recall that if an A operator is invertible modulo some ideal $\mathfrak{J}$ and the limit operators for all operators in this ideal vanish, then the limit operator of A is invertible whenever it exists. Further we calculate the limit operators of the operator N with respect to two different systems of pseudoisometries (dilations and modulations). Let $\mathcal{K}$ be the ideal of all compact operators on $L^p(\mathbb{R}_+)$. In [6] we introduced the algebra $\mathcal{Z}$ generated by the ideal $\mathcal{K}$, the operators I, S, and cR, where $c \in SO(\mathbb{R}_+)$ and R is the operator with fixed singularities at 0 and ∞ given by

$$(Rf)(t) := \frac{1}{\pi i} \int_0^\infty \frac{f(\tau)}{\tau + t} \quad (t \in \mathbb{R}_+).$$

It turns out that the algebra Λ of all operators commuting with the elements of $\mathcal{Z}$ modulo the ideal $\mathcal{K}$ contains the operator N. In Section 5 we state a consequence of the Allan-Douglas local principle for $A \in \Lambda$, which was obtained in [6]. In Section 6 we formulate an invertibility criterion for $aI - bW_\alpha$ with slowly oscillating data (Theorem 6.1) and prove two auxiliary statements: a corollary of Theorem 6.1 related to the existence of infinite dimensional kernel or cokernel for $aI - bW_\alpha$ and a criterion for the invertibility of $aI - bW_\alpha$ with multiplicative shift α. In the proof of the latter result we use limit operators with respect to a specially chosen system of modulations, so that the limit operator of W_α is equal to W_α. Section 7 is devoted to the proof of Theorem 1.2. First we observe that the limit operators with respect to dilations are

$$\big(a(\xi)I - b(\xi)W_{\alpha_\xi}\big)P_+ + \big(c(\xi)I - d(\xi)W_{\alpha_\xi}\big)P_-, \tag{1.3}$$

where $\xi \in \Delta$ and $\alpha_\xi(t) = e^{\omega(\xi)}t$ is a multiplicative shift. Since N is Fredholm, all limit operators are invertible for $\xi \in \Delta$. Applying the results of Sections 5 and 6, we prove that then the operators $a(\xi)I - b(\xi)W_{\alpha_\xi}$ and $c(\xi)I - d(\xi)W_{\alpha_\xi}$ are invertible for all $\xi \in \Delta$. Since the fibers $M_0(SO(\mathbb{R}_+))$ and $M_\infty(SO(\mathbb{R}_+))$ are connected, from the above observation and Theorem 6.1 it follows that the

operators $aI - bW_\alpha$ and $cI - dW_\alpha$ are invertible, and this is condition (i) of Theorem 1.1. On the other hand, the (invertible) limit operators (1.3) are similar to the Mellin convolution operators with the semi-almost periodic symbols n_ξ. Applying the invertibility criterion for such operators (Theorem 3.5), we arrive at condition (ii) of Theorem 1.1.

2. Slowly oscillating functions and shifts

2.1. Properties of slowly oscillating functions. The following two lemmas give important properties of the fibers $M_0(SO(\mathbb{R}_+))$ and $M_\infty(SO(\mathbb{R}_+))$.

Lemma 2.1 ([8, Proposition 2.2])**.** *Let $\{a_k\}_{k=1}^\infty$ be a countable subset of the space $SO(\mathbb{R}_+)$ and $s \in \{0, \infty\}$. For each $\xi \in M_s(SO(\mathbb{R}_+))$ there exists a sequence $\{t_n\} \subset \mathbb{R}_+$ such that $t_n \to s$ as $n \to \infty$ and*

$$\xi(a_k) = \lim_{n\to\infty} a_k(t_n) \quad \text{for all} \quad k \in \mathbb{N}. \tag{2.1}$$

Conversely, if $\{t_n\} \subset \mathbb{R}_+$ is a sequence such that $t_n \to s$ as $n \to \infty$, then there exists a functional $\xi \in M_s(SO(\mathbb{R}_+))$ such that (2.1) holds.

Lemma 2.2. *The fibers $M_0(SO(\mathbb{R}_+))$ and $M_\infty(SO(\mathbb{R}_+))$ are connected compact Hausdorff spaces.*

Proof. Fix $s \in \{0, \infty\}$. Since $M_s(SO(\mathbb{R}_+))$ is a closed subset of the compact Hausdorff space $M(SO(\mathbb{R}_+))$, we conclude that $M_s(SO(\mathbb{R}_+))$ also is a compact Hausdorff space. Suppose the fiber $M_s(SO(\mathbb{R}_+))$ is disconnected. Then there exist two disjoint closed subsets X_1 and X_2 such that $M_s(SO(\mathbb{R}_+)) = X_1 \cup X_2$. Take a continuous function $\widehat{a}$ on $M_s(SO(\mathbb{R}_+))$ such that $\widehat{a}(X_1) \subset [0, 1/3]$ and $\widehat{a}(X_2) \subset [2/3, 1]$. By the Tietze extension theorem (see e.g. [13, Theorem IV.11]), the function $\widehat{a}$ is extended to a continuous function on the whole compact space $M(SO(\mathbb{R}_+))$. We denote this extension again by $\widehat{a}$. Because $SO(\mathbb{R}_+)$ is a C^*-algebra, the function $\widehat{a} \in C(M(SO(\mathbb{R}_+)))$ is the Gelfand transform of a function $a \in SO(\mathbb{R}_+)$. Then in view of Lemma 2.1 there are sequences $t_n', t_n'' \to s$ such that there exist $\lim_{n\to\infty} a(t_n') \in [0, 1/3]$ and $\lim_{n\to\infty} a(t_n'') \in [2/3, 1]$. Since $a \in SO(\mathbb{R}_+)$ is continuous on $\mathbb{R}_+$, there are points t_n between t_n' and t_n'' such that $a(t_n) = 1/2$. Then $t_n \to s$, $\lim_{n\to\infty} a(t_n) = 1/2$, and hence $1/2 \in \widehat{a}(X_1) \cup \widehat{a}(X_2)$, a contradiction. Thus, $M_s(SO(\mathbb{R}_+))$ is a connected set. $\square$

Repeating literally the proofs of [5, Proposition 3.3] and [5, Lemma 3.5], we obtain the following two statements.

Lemma 2.3. *Suppose $\varphi \in C^1(\mathbb{R}_+)$ and put $\psi(t) := t\varphi'(t)$ for $t \in \mathbb{R}_+$. If $\varphi, \psi \in SO(\mathbb{R}_+)$, then $\lim_{t\to s} \psi(t) = 0$ for $s \in \{0, \infty\}$.*

Lemma 2.4. *Let $a \in SO(\mathbb{R}_+)$. Suppose continuous functions $f_j : \mathbb{R}_+ \to \mathbb{R}_+$ $(j = 1, 2)$ and $\mathcal{F} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfy the relation*

$$xf_1(y) \le \mathcal{F}(x, y) \le xf_2(y), \quad x, y \in \mathbb{R}_+.$$

If for some sequence $t = \{t_n\}_{n=1}^{\infty}$ tending to $s \in \{0, \infty\}$ the limit

$$\lim_{n\to\infty} a(t_n) =: a_t$$

exists, then for every $y \in \mathbb{R}_+$ the limit $\lim_{n\to\infty} a(\mathcal{F}(t_n, y))$ also exists. Moreover,

$$\lim_{n\to\infty} a(\mathcal{F}(t_n, y)) = a_t,$$

and the convergence is uniform on every segment $J \subset \mathbb{R}_+$.

2.2. Properties of slowly oscillating shifts. In this subsection we list necessary properties of slowly oscillating shifts.

Lemma 2.5 ([6, Lemma 2.2])**.** *An orientation-preserving non-Carleman shift $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to $SOS(\mathbb{R}_+)$ if and only if*

$$\alpha(t) = te^{\omega(t)}, \quad t \in \mathbb{R}_+, \tag{2.2}$$

for some real-valued function $\omega \in SO(\mathbb{R}_+) \cap C^1(\mathbb{R}_+)$ such that the function $t \mapsto t\omega'(t)$ also belongs to $SO(\mathbb{R}_+)$ and $\inf_{t\in\mathbb{R}_+} \big(1 + t\omega'(t)\big) > 0$.

The function ω in (2.2) is referred to as the exponent function of α.

Lemma 2.6 ([6, Lemma 2.3])**.** *Suppose $c \in SO(\mathbb{R}_+)$ and $\alpha \in SOS(\mathbb{R}_+)$. Then $c \circ \alpha \in SO(\mathbb{R}_+)$.*

Lemma 2.7 ([6, Lemma 2.4])**.** *If $\alpha \in SOS(\mathbb{R}_+)$, then $\beta \in SOS(\mathbb{R}_+)$.*

Lemma 2.8. *Suppose $\alpha \in SOS(\mathbb{R}_+)$ and ω is the exponent function of α. Then $\alpha'(\xi) = e^{\omega(\xi)}$ for all $\xi \in \Delta$.*

Proof. By Lemma 2.5, $\alpha(\tau) = \tau e^{\omega(\tau)}$ for $\tau \in \mathbb{R}_+$ with $\omega \in SO(\mathbb{R}_+) \cap C^1(\mathbb{R}_+)$ and $\psi(\tau) = \tau\omega'(\tau) \in SO(\mathbb{R}_+)$. From Lemma 2.3 it follows that

$$\lim_{\tau\to s} \tau\omega'(\tau) = 0 \quad \text{for} \quad s \in \{0, \infty\}. \tag{2.3}$$

Fix $s \in \{0, \infty\}$. By Lemma 2.1, for a given $\xi \in M_s(SO(\mathbb{R}_+))$ there exists a sequence $t_n \to s$ as $n \to \infty$ such that

$$\omega(\xi) = \lim_{n\to\infty} \omega(t_n), \quad \alpha'(\xi) = \lim_{n\to\infty} \alpha'(t_n) \tag{2.4}$$

(recall that $\alpha' \in SO(\mathbb{R}_+)$). Clearly, $\alpha'(\tau) = (1 + t\omega'(\tau))e^{\omega(\tau)}$ for $\tau \in \mathbb{R}_+$. Combining this relation with (2.3)–(2.4), we get $\alpha'(\xi) = e^{\omega(\xi)}$. ☐

3. Convolution operators

3.1. Fourier convolution operators. For a Banach space X, let $\mathcal{B}(X)$ be the Banach algebra of all bounded linear operators on X and let $\mathcal{K}(X)$ be the closed two-sided ideal of all compact operators on X.

Let $F : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ denote the Fourier transform,

$$(Ff)(x) := \int_{\mathbb{R}} f(y)e^{-ixy}dy \quad (x \in \mathbb{R}),$$

and let $F^{-1} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the inverse of F. A function $a \in L^\infty(\mathbb{R})$ is called a Fourier multiplier if the map $f \mapsto F^{-1}aFf$ maps $L^2(\mathbb{R}) \cap L^p(\mathbb{R})$ onto itself and extends to a bounded operator on $L^p(\mathbb{R})$. The latter operator is then denoted by $W^0(a)$. We let $M_p(\mathbb{R})$ stand for the set of all Fourier multipliers on $L^p(\mathbb{R})$. One can show that $M_p(\mathbb{R})$ is a Banach algebra under the norm

$$\|a\|_{M_p(\mathbb{R})} := \|W^0(a)\|_{\mathcal{B}(L^p(\mathbb{R}))}.$$

3.2. Mellin convolution operators. Let $d\mu(t) = dt/t$ be the (normalized) invariant measure on $\mathbb{R}_+$. Consider the Fourier transform on $L^2(\mathbb{R}_+, d\mu)$, which is usually referred to as the Mellin transform and is defined by

$$M : L^2(\mathbb{R}_+, d\mu) \to L^2(\mathbb{R}), \quad (Mf)(x) = \int_{\mathbb{R}_+} f(t)t^{-ix}\,\frac{dt}{t}.$$

It is an invertible operator, with inverse given by

$$M^{-1} : L^2(\mathbb{R}) \to L^2(\mathbb{R}_+, d\mu), \quad (M^{-1}g)(t) = \frac{1}{2\pi}\int_{\mathbb{R}} g(x)t^{ix}\,dx.$$

Let E be the isometric isomorphism

$$E : L^p(\mathbb{R}_+, d\mu) \to L^p(\mathbb{R}), \quad (Ef)(x) := f(e^x) \quad (x \in \mathbb{R}). \tag{3.1}$$

Then the map $A \mapsto E^{-1}AE$ transforms the Fourier convolution operator given by $W^0(a) = F^{-1}aF$ to the Mellin convolution operator

$$\mathrm{Co}(a) := M^{-1}aM$$

with the same symbol a. Hence the class of Fourier multipliers on $L^p(\mathbb{R})$ coincides with the class of Mellin multipliers on $L^p(\mathbb{R}_+, d\mu)$.

3.3. Piecewise continuous multipliers. We denote by PC the C^*-algebra of all bounded piecewise continuous functions on $\dot{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. By definition, $a \in PC$ if and only if $a \in L^\infty(\mathbb{R})$ and the one-sided limits

$$a(x_0 - 0) := \lim_{x \to x_0 - 0} a(x), \quad a(x_0 + 0) := \lim_{x \to x_0 + 0} a(x)$$

exist for each $x_0 \in \dot{\mathbb{R}}$. If a function a is given everywhere on $\mathbb{R}$, then its total variation of a is defined by

$$V(a) := \sup \sum_{k=1}^{n} |a(x_k) - a(x_{k-1})|,$$

where the supremum is taken over all $n \in \mathbb{N}$ and

$$-\infty < x_0 < x_1 < \cdots < x_n < +\infty.$$

If a has a finite total variation, then it has finite one-sided limits $a(x-0)$ and $a(x+0)$ for all $x \in \dot{\mathbb{R}}$, that is, $a \in PC$. If a is an absolutely continuous function of finite total variation on $\mathbb{R}$, then $a' \in L^1(\mathbb{R})$ and

$$V(a) = \int_{\mathbb{R}} |a'(x)| dx$$

(see, e.g., [11, Chap. VIII, Sections 3 and 9; Chap. XI, Section 4]).

The following theorem gives an important subset of $M_p(\mathbb{R})$. Its proof can be found, e.g., in [2, Theorem 17.1].

Theorem 3.1 (Stechkin's inequality)**.** *If $a \in PC$ has finite total variation $V(a)$, then $a \in M_p(\mathbb{R})$ and*

$$\|a\|_{M_p(\mathbb{R})} \le \|S_{\mathbb{R}}\|_{\mathcal{B}(L^p(\mathbb{R}))} \big(\|a\|_{L^\infty(\mathbb{R})} + V(a)\big),$$

where $S_{\mathbb{R}}$ is the Cauchy singular integral operator on $\mathbb{R}$.

According to [2, p. 325], let PC_p be the closure in $M_p(\mathbb{R})$ of the set of all functions $a \in PC$ with finite total variation on $\mathbb{R}$. Following [2, p. 331], put

$$C_p(\overline{\mathbb{R}}) := PC_p \cap C(\mathbb{R}), \quad \overline{\mathbb{R}} := [-\infty, +\infty].$$

3.4. Algebra generated by the Cauchy singular integral operator. Suppose $\mathfrak{A}$ is a Banach algebra and $\mathfrak{S}$ is a subset of $\mathfrak{A}$. Let $\operatorname{alg}_{\mathfrak{A}} \mathfrak{S}$ denote the smallest closed subalgebra of $\mathfrak{A}$ containing $\mathfrak{S}$ and let $\operatorname{id}_{\mathfrak{A}} \mathfrak{S}$ denote the smallest closed two-sided ideal of $\mathfrak{A}$ containing $\mathfrak{S}$.

Let $\mathcal{B} := \mathcal{B}(L^p(\mathbb{R}_+))$, $\mathcal{K} := \mathcal{K}(L^p(\mathbb{R}_+))$, and $\mathcal{A} := \operatorname{alg}_{\mathcal{B}}\{I, S\}$. Consider the isometric isomorphism

$$\Phi : L^p(\mathbb{R}_+) \to L^p(\mathbb{R}_+, d\mu), \quad (\Phi f)(t) := t^{1/p} f(t) \quad (t \in \mathbb{R}_+). \tag{3.2}$$

The following fact is well known (see, e.g., [14, Section 2]).

Theorem 3.2. *The algebra $\mathcal{A}$ is the smallest closed subalgebra of $\mathcal{B}$ that contains the operators $\Phi^{-1}\operatorname{Co}(a)\Phi$ with $a \in C_p(\overline{\mathbb{R}})$. The functions*

$$s_p(x) := \coth[\pi(x + i/p)] \quad r_p(x) := 1/\sinh[\pi(x + i/p)] \quad (x \in \mathbb{R})$$

belong to $C_p(\overline{\mathbb{R}})$ and the operators S and R are similar to the Mellin convolution operators:

$$\Phi S \Phi^{-1} = \operatorname{Co}(s_p), \quad \Phi R \Phi^{-1} = \operatorname{Co}(r_p).$$

From $s_p^2 - r_p^2 = 1$ and Theorem 3.2 it follows that

$$4P_+P_- = 4P_-P_+ = I - S^2 = -R^2. \tag{3.3}$$

Theorem 3.3 ([6, Corollary 6.4])**.** *If $a \in SO(\mathbb{R}_+)$ and $\alpha \in SOS(\mathbb{R}_+)$, then for every $A \in \mathcal{A}$ the operators $aA - AaI$ and $W_\alpha A - AW_\alpha$ are compact.*

3.5. Semi-almost periodic multipliers. The following simple statement motivates us to enlarge the class of piecewise continuous multipliers.

Lemma 3.4. *Let $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ be a multiplicative shift given by $\alpha(t) = kt$ for all $t \in \mathbb{R}_+$ with some $k \in \mathbb{R}_+$. Then $\Phi W_\alpha \Phi^{-1} = \operatorname{Co}(m)$ with $m(x) := e^{i(x+i/p)\log k}$ for $x \in \mathbb{R}$.*

Proof. The proof is a matter of a direct calculation. ☐

A function $p : \mathbb{R} \to \mathbb{C}$ of the form $p(x) = \sum_{\lambda\in\Omega} r_\lambda e^{i\lambda x}$, where $r_\lambda \in \mathbb{C}$, $\lambda \in \mathbb{R}$, and Ω is a finite subset of $\mathbb{R}$, is called an almost periodic polynomial. The set of all almost periodic polynomials is denoted by AP_0. From Lemma 3.4 it follows that $AP_0 \subset M_p(\mathbb{R})$. According to [2, p. 372], AP_p denotes the closure of the set of all almost periodic polynomials in the norm of $M_p(\mathbb{R})$ and SAP_p denotes the smallest closed subalgebra of $M_p(\mathbb{R})$ that contains $C_p(\overline{\mathbb{R}})$ and AP_p.

Applying the inverse closedness of the algebra SAP_p in $L^\infty(\mathbb{R})$ (see [2, Proposition 19.4]), we immediately get the following.

Theorem 3.5. *Suppose $a \in SAP_p$. The Mellin convolution operator $\operatorname{Co}(a)$ is invertible on the space $L^p(\mathbb{R}_+, d\mu)$ if and only if $\inf_{x\in\mathbb{R}} |a(x)| > 0$.*

4. Limit operators

4.1. Abstract approach. In our previous work [5] the techniques of limit operators (see, e.g., [1, 10, 12]) was successfully used to study the invertibility of binomial functional operators that are now the coefficients of the singular integral operator with shift N given by (1.1). In what follows we make use of such techniques to obtain a necessary condition for the Fredholmness of the operator N. Let us recall the abstract version of such techniques.

Let X be a Banach space and let X^* be its dual space. We say that an operator $U \in \mathcal{B}(X)$ is a *pseudoisometry* if U is invertible in $\mathcal{B}(X)$ and

$$\|U\|_{\mathcal{B}(X)} = 1/\|U^{-1}\|_{\mathcal{B}(X)}.$$

Let $A \in \mathcal{B}(X)$ and $\mathcal{U} = \{U_n\}_{n=1}^{\infty}$ be a sequence of pseudoisometries. If the strong limits

$$A_{\mathcal{U}} := \operatorname*{s-lim}_{n\to\infty}(U_n^{-1}AU_n) \text{ in } \mathcal{B}(X), \quad A_{\mathcal{U}^*} := \operatorname*{s-lim}_{n\to\infty}(U_n^{-1}AU_n)^* \text{ in } \mathcal{B}(X^*) \tag{4.1}$$

exist, then always $(A_{\mathcal{U}})^* = A_{\mathcal{U}^*}$, and we will refer the operator $A_{\mathcal{U}}$ to as a *limit operator* for the operator A with respect to the sequence $\mathcal{U}$. Note that usually the limit operator $A_{\mathcal{U}}$ is defined independently of the existence of the strong limit $A_{\mathcal{U}^*}$ (see, e.g., [1, 12]), while we need the existence of the both limits (4.1) for our purposes. If the limit operator $A_{\mathcal{U}}$ exists, then it is uniquely determined by A and $\mathcal{U}$, which justifies the notation $A_{\mathcal{U}}$.

In the next statement we collect basic properties of limit operators.

Lemma 4.1. *Suppose $\mathcal{U} = \{U_n\}_{n=1}^{\infty} \subset \mathcal{B}(X)$ is a sequence of pseudoisometries.*

(a) *If $A \in \mathcal{B}(X)$ and $A_{\mathcal{U}}$ exists, then $\|A_{\mathcal{U}}\|_{\mathcal{B}(X)} \le \|A\|_{\mathcal{B}(X)}$.*

(b) *If $A, B \in \mathcal{B}(X)$, $\alpha \in \mathbb{C}$, and if the limit operators $A_{\mathcal{U}}$, $B_{\mathcal{U}}$ exist, then the limit operators $(\alpha A)_{\mathcal{U}}$, $(A+B)_{\mathcal{U}}$, $(AB)_{\mathcal{U}}$ also exist and*

$$(\alpha A)_{\mathcal{U}} = \alpha A_{\mathcal{U}}, \quad (A+B)_{\mathcal{U}} = A_{\mathcal{U}} + B_{\mathcal{U}}, \quad (AB)_{\mathcal{U}} = A_{\mathcal{U}} B_{\mathcal{U}}.$$

(c) *If $A \in \mathcal{B}(X)$ and if $\{A_m\}_{m=1}^{\infty} \subset \mathcal{B}(X)$ is such that the limit operators $(A_m)_{\mathcal{U}}$ exist for all $m \in \mathbb{N}$ and $\|A - A_m\|_{\mathcal{B}(X)} \to 0$ as $m \to \infty$, then the limit operator $A_{\mathcal{U}}$ exists and $\|A_{\mathcal{U}} - (A_m)_{\mathcal{U}}\|_{\mathcal{B}(X)} \to 0$ as $m \to \infty$.*

The proofs of the above results can be found in [10, Proposition 3.4] or [12, Proposition 1.2.2].

Theorem 4.2. *Let X be a Banach space, let $\mathfrak{A}$ be a closed subalgebra of $\mathcal{B}(X)$, and let $\mathfrak{J}$ be a closed two-sided ideal of $\mathfrak{A}$. Suppose $A \in \mathfrak{A}$ and $\mathcal{U} = \{U_n\}_{n=1}^{\infty} \subset \mathcal{B}(X)$ is a sequence of pseudoisometries such that the limit operator $A_{\mathcal{U}}$ exists and the limit operators $J_{\mathcal{U}}$ exist and are equal to zero for all $J \in \mathfrak{J}$. If the coset $A + \mathfrak{J}$ is invertible in the quotient algebra $\mathfrak{A}/\mathfrak{J}$, then the limit operator $A_{\mathcal{U}}$ is invertible.*

The proof is developed by analogy with [12, Proposition 1.2.9].

4.2. Strong convergence of shift operators. To calculate limit operators for the shift operator W_α, we need a result on the strong convergence of shift operators.

Lemma 4.3. *Let $\alpha_n : \mathbb{R}_+ \to \mathbb{R}_+$ for $n \in \mathbb{N} \cup \{0\}$ be orientation-preserving diffeomorphisms having only two fixed points 0 and ∞, and β_n be their inverses. If $\log \alpha_n' \in L^\infty(\mathbb{R}_+)$ for all $n \in \mathbb{N} \cup \{0\}$ and*

(i) $\sup\limits_{n\in\mathbb{N}\cup\{0\}} \|\beta_n'\|_{L^\infty(\mathbb{R}_+)} < \infty$,

(ii) *$\alpha_n \to \alpha_0$ pointwise on $\mathbb{R}_+$ as $n \to \infty$;*

then the sequence of shift operators $W_{\alpha_n} \in \mathcal{B}$ converges strongly to the shift operator $W_{\alpha_0} \in \mathcal{B}$.

Proof. The idea of the proof is borrowed from [4, Theorem 1]. Let χ_E denote the characteristic function of a set $E \subset \mathbb{R}_+$. Since the linear span of the set $\{\chi_{[0,\tau]} : \tau \in \mathbb{R}_+\}$ is dense in the space $L^p(\mathbb{R}_+)$ and the operators W_{α_n} are uniformly bounded on $L^p(\mathbb{R}_+)$ in view of (i), it is sufficient to prove that

$$\lim_{n\to\infty} \left\| W_{\alpha_n}\chi_{[0,\tau]} - W_{\alpha_0}\chi_{[0,\tau]} \right\|_{L^p(\mathbb{R}_+)} = 0 \quad \text{for all} \quad \tau \in \mathbb{R}_+. \tag{4.2}$$

It is easy to see that

$$\begin{aligned} \left\| W_{\alpha_n}\chi_{[0,\tau]} - W_{\alpha_0}\chi_{[0,\tau]} \right\|^p_{L^p(\mathbb{R}_+)} &= \int_{\mathbb{R}_+} \left| \chi_{[0,\tau]}(\alpha_n(t)) - \chi_{[0,\tau]}(\alpha_0(t)) \right|^p dt \\ &= \int_{\mathbb{R}_+} \left| \chi_{[0,\beta_n(\tau)]}(t) - \chi_{[0,\beta_0(\tau)]}(t) \right|^p dt = \left| \beta_n(\tau) - \beta_0(\tau) \right|. \end{aligned} \tag{4.3}$$

On the other hand,

$$\begin{aligned} \left| \beta_n(\tau) - \beta_0(\tau) \right| &= \Big| \beta_n\big[\alpha_0(\beta_0(\tau))\big] - \beta_n\big[\alpha_n(\beta_0(\tau))\big] \Big| \\ &\le \sup_{n\in\mathbb{N}} \left\| \beta_n' \right\|_{L^\infty(\mathbb{R}_+)} \left| \alpha_0(\beta_0(\tau)) - \alpha_n(\beta_0(\tau)) \right|. \end{aligned} \tag{4.4}$$

From (4.4) and the hypotheses of the lemma it follows that

$$|\beta_n(\tau) - \beta_0(\tau)| = o(1) \quad \text{as} \quad n \to \infty$$

for every $\tau \in \mathbb{R}_+$. Combining this with (4.3), we arrive at (4.2). $\square$

4.3. Realization with dilations. For $x \in \mathbb{R}_+$, consider the dilation operator V_x defined on $L^p(\mathbb{R}_+)$ by

$$(V_x f)(t) := f(t/x) \quad (t \in \mathbb{R}_+).$$

It is easy to see that V_x is invertible on the space $L^p(\mathbb{R}_+)$ and $V_x^{-1} = V_{1/x}$. Moreover, $\|V_x\|_{\mathcal{B}} = x^{1/p}$ and hence V_x is a pseudoisometry for every $x \in \mathbb{R}_+$.

Fix $s \in \{0, \infty\}$. We say that a sequence $h := \{h_n\}_{n=1}^\infty \subset \mathbb{R}_+$ is a test sequence relative to the point s if

$$\lim_{n\to\infty} h_n = s.$$

With each test sequence h relative to the point s we associate the sequence of pseudoisometries $\mathcal{V}_h^s := \{V_{h_n}\}_{n=1}^\infty \subset \mathcal{B}$.

Lemma 4.4. *Let $h := \{h_n\}_{n=1}^\infty \subset \mathbb{R}_+$ be a test sequence relative to $s \in \{0, \infty\}$. For any operator $K \in \mathcal{K}$, the limit operator $K_{\mathcal{V}_h^s}$ with respect to the sequence of pseudoisometries $\mathcal{V}_h^s := \{V_{h_n}\}_{n=1}^\infty \subset \mathcal{B}$ exists and is the zero operator.*

Proof. Consider the isometric isomorphism $E\Phi : L^p(\mathbb{R}_+) \to L^p(\mathbb{R})$, where E is defined by (3.1) and Φ is defined by (3.2). Then $\widetilde{K} := E\Phi K \Phi^{-1} E^{-1}$ is compact on $L^p(\mathbb{R})$ for every $K \in \mathcal{K}(L^p(\mathbb{R}_+))$, and for every $x \in \mathbb{R}_+$, $E\Phi V_x \Phi^{-1} E^{-1} = \widetilde{V}_x$, where $\widetilde{V}_x \in \mathcal{B}(L^p(\mathbb{R}))$ and $(\widetilde{V}_x f)(y) = x^{1/p} f(y - \log x)$ for all $y \in \mathbb{R}$. By [2, Lemma 18.9], $\operatorname*{s-lim}_{n\to\infty} \widetilde{V}_{h_n}^{-1} \widetilde{K} \widetilde{V}_{h_n} I = 0$ on $L^p(\mathbb{R})$ for every test sequence $h = \{h_n\}_{n=1}^\infty \subset \mathbb{R}_+$. Therefore

$$\operatorname*{s-lim}_{n\to\infty} V_{h_n}^{-1} K V_{h_n} = \operatorname*{s-lim}_{n\to\infty} \Phi^{-1} E^{-1} \widetilde{V}_{h_n}^{-1} \widetilde{K} \widetilde{V}_{h_n} E\Phi = 0 \quad \text{on} \quad L^p(\mathbb{R}_+).$$

Analogously, $(V_{h_n}^{-1} K V_{h_n})^* = V_{h_n}^{-1} K^* V_{h_n}$ converges strongly to zero on the space $L^q(\mathbb{R}_+)$. Thus, the limit operator $K_{\mathcal{V}_h^s}$ exists and is equal to zero. $\square$

Lemma 4.5. *Suppose $a, b, c, d \in SO(\mathbb{R}_+)$, $\alpha \in SOS(\mathbb{R}_+)$, and the operator N is given by* (1.1)*. Let $s \in \{0, \infty\}$. For every functional $\xi \in M_s(SO(\mathbb{R}_+))$ there exists a test sequence $h^\xi = \{h_n^\xi\}_{n=1}^\infty \subset \mathbb{R}_+$ relative to the point s such that the limit operator $N_{\mathcal{V}_{h^\xi}^s}$ with respect to the sequence of pseudoisometries $\mathcal{V}_{h^\xi}^s := \{V_{h_n^\xi}\}_{n=1}^\infty \subset \mathcal{B}$ exists and*

$$N_{\mathcal{V}_{h^\xi}^s} = \big(a(\xi)I - b(\xi)W_{\alpha_\xi}\big)P_+ + \big(c(\xi)I - d(\xi)W_{\alpha_\xi}\big)P_-, \tag{4.5}$$

where $\alpha_\xi(t) := e^{\omega(\xi)} t$ and $\omega(t) := \log[\alpha(t)/t]$ for $t \in \mathbb{R}_+$.

Proof. Fix $s \in \{0, \infty\}$ and $\xi \in M_s(SO(\mathbb{R}_+))$. From Lemma 2.7 it follows that $\beta := \alpha_{-1}$ is a slowly oscillating shift. Then, by Lemma 2.5, $\alpha', \beta' \in SO(\mathbb{R}_+)$ and the functions ω and $\zeta(t) := \log[\beta(t)/t]$ are real-valued slowly oscillating functions. Clearly, $\overline{a}, \overline{b}, \overline{c}, \overline{d} \in SO(\mathbb{R}_+)$. Let

$$\mathcal{G} := \{a, b, c, d, \alpha', \beta', \omega, \zeta\} \subset SO(\mathbb{R}_+).$$

By Lemma 2.1, there exists a test sequence $h^\xi = \{h_n^\xi\}_{n=1}^\infty \subset \mathbb{R}_+$ relative to the point s such that the limit

$$g(\xi) = \xi(g) = \lim_{n\to\infty} g(h_n^\xi) \tag{4.6}$$

exists for every function $g \in \mathcal{G}$. Lemma 2.4 implies that for every $t \in \mathbb{R}_+$,

$$\lim_{n\to\infty} |g(h_n^\xi t) - g(h_n^\xi)| = 0, \tag{4.7}$$

and the convergence is uniform in t on every segment $J \subset \mathbb{R}_+$. Since

$$V_{h_n^\xi}^{-1}(gI)V_{h_n^\xi} = g_n I, \quad (V_{h_n^\xi}(gI)V_{h_n^\xi})^* = \overline{g_n} I,$$

where $g_n(t) := g(h_n^\xi t)$ for all $t \in \mathbb{R}_+$, we infer from (4.6) and (4.7) that for the multiplication operator on $L^p(\mathbb{R}_+)$ and its adjoint on $L^q(\mathbb{R}_+)$,

$$\begin{aligned}
\operatorname*{s-lim}_{n\to\infty} V_{h_n^\xi}^{-1}(gI)V_{h_n^\xi} &= \operatorname*{s-lim}_{n\to\infty} g_n I = \lim_{n\to\infty} g(h_n^\xi)I = g(\xi)I,\\
\operatorname*{s-lim}_{n\to\infty}(V_{h_n^\xi}^{-1}(gI)V_{h_n^\xi})^* &= \operatorname*{s-lim}_{n\to\infty} \overline{g_n} I = \lim_{n\to\infty} \overline{g(h_n^\xi)}I = \overline{g(\xi)}I.
\end{aligned}$$

Hence

$$(gI)_{\mathcal{V}^s_{h^\xi}} = g(\xi)I \quad \text{for} \quad g \in \{a,b,c,d\}. \tag{4.8}$$

From Lemma 2.5 it follows that $\alpha(t) = te^{\omega(t)}$. Therefore, for all $n \in \mathbb{N}$,

$$V_{h_n^\xi}^{-1} W_\alpha V_{h_n^\xi} = W_{\alpha_\xi^{(n)}}, \tag{4.9}$$

where $\alpha_\xi^{(n)}(t) := te^{\omega(h_n^\xi t)}$ for $t \in \mathbb{R}_+$. From (4.6)–(4.7) we conclude that

$$\lim_{n\to\infty} \alpha_\xi^{(n)}(t) = te^{\omega(\xi)} = \alpha_\xi(t), \quad t \in \mathbb{R}_+. \tag{4.10}$$

Since $\log \alpha'$ is bounded, we have

$$0 < m_\alpha := \inf_{y\in\mathbb{R}_+} \alpha'(y). \tag{4.11}$$

Let $\beta_\xi^{(n)}$ be the inverse shift to $\alpha_\xi^{(n)}$. It is easy to see that for all $n \in \mathbb{N}$ and $t \in \mathbb{R}_+$,

$$(\beta_\xi^{(n)})'(t) = \frac{1}{(\alpha_\xi^{(n)})'[\beta_\xi^{(n)}(t)]} = \frac{1}{\alpha'[h_n^\xi \beta_\xi^{(n)}(t)]}.$$

From this equality and (4.11) it follows that for all $n \in \mathbb{N}$ and $t \in \mathbb{R}_+$,

$$(\beta_\xi^{(n)})'(t) \le 1/m_\alpha < +\infty. \tag{4.12}$$

Moreover, the derivative of the inverse shift to α_ξ is constant. Thus, combining (4.9)–(4.12) and Lemma 4.3, we see that

$$\operatorname*{s-lim}_{n\to\infty} V_{h_n^\xi}^{-1} W_\alpha V_{h_n^\xi} = W_{\alpha_\xi} \quad \text{on} \quad L^p(\mathbb{R}_+). \tag{4.13}$$

It is easy to see that $(W_\alpha)^* = \beta' W_\beta$. We have already proved that

$$\operatorname*{s-lim}_{n\to\infty} V_{h_n^\xi}^{-1}(\beta' I) V_{h_n^\xi} = \beta'(\xi) I \quad \text{on} \quad L^q(\mathbb{R}_+). \tag{4.14}$$

Analogously to (4.13) one can show that

$$\operatorname*{s-lim}_{n\to\infty} V_{h_n^\xi}^{-1} W_\beta V_{h_n^\xi} = W_{\beta_\xi} \quad \text{on} \quad L^q(\mathbb{R}_+), \tag{4.15}$$

where $\beta_\xi(t) = te^{\zeta(\xi)}$. From (4.14)–(4.15) we obtain

$$\begin{aligned}\operatorname*{s-lim}_{n\to\infty}\left(V_{h_n^\xi}^{-1}W_\alpha V_{h_n^\xi}\right)^* &= \left(\operatorname*{s-lim}_{n\to\infty}V_{h_n^\xi}^{-1}(\beta' I)V_{h_n^\xi}\right)\left(\operatorname*{s-lim}_{n\to\infty}V_{h_n^\xi}^{-1}W_\beta V_{h_n^\xi}\right)\\ &= \beta'(\xi)W_{\beta_\xi}. \end{aligned}\tag{4.16}$$

Equalities (4.13) and (4.16) imply that

$$(W_\alpha)_{\mathcal{V}^s_{h^\xi}} = W_{\alpha_\xi}. \tag{4.17}$$

It is easy to see that $S^* = S$ and $V_{h_n^\xi}^{-1}SV_{h_n^\xi} = S$. Hence

$$(P_+)_{\mathcal{V}_h^\xi} = P_+, \quad (P_-)_{\mathcal{V}_h^\xi} = P_-. \tag{4.18}$$

Combining (4.8), (4.17), and (4.18) with Lemma 4.1(b), we see that the limit operator $N_{\mathcal{V}^s_{h^\xi}}$ exists and is calculated by (4.5). ☐

4.4. Realization with modulations. For $x \in \mathbb{R}$, consider the modulation operator E_x defined on $L^p(\mathbb{R}_+)$ by

$$(E_xf)(t) := t^{ix}f(t) \quad (t \in \mathbb{R}_+).$$

It is clear that E_x is invertible on the space $L^p(\mathbb{R}_+)$ and $E_x^{-1} = E_{-x}$. Moreover, $\|E_x\|_{\mathcal{B}} = 1$ and hence E_x is a pseudoisometry for each $x \in \mathbb{R}$.

Fix $s \in \{-\infty, +\infty\}$. We say that a sequence $\mu := \{\mu_n\}_{n=1}^\infty \subset \mathbb{R}$ is a test sequence relative to the point s if

$$\lim_{n\to\infty}\mu_n = s.$$

With each test sequence μ relative to the point s we associate the sequence of pseudoisometries $\mathcal{E}^s_\mu := \{E_{\mu_n}\}_{n=1}^\infty \subset \mathcal{B}$.

Lemma 4.6. *Suppose $\mu := \{\mu_n\}_{n=1}^\infty \subset \mathbb{R}$ is a test sequence relative to a point $s \in \{-\infty, +\infty\}$. For any operator $K \in \mathcal{K}$, the limit operator $K_{\mathcal{E}^s_\mu}$ with respect to the sequence of pseudoisometries $\mathcal{E}^s_\mu := \{E_{\mu_n}\}_{n=1}^\infty \subset \mathcal{B}$ exists and is the zero operator.*

Proof. Following the proof of Lemma 4.4, consider the isometric isomorphism $E\Phi : L^p(\mathbb{R}_+) \to L^p(\mathbb{R})$ and the operator $\widetilde{K} := E\Phi K\Phi^{-1}E^{-1} \in \mathcal{K}(L^p(\mathbb{R}))$, where $K \in \mathcal{K}(L^p(\mathbb{R}_+))$. For every $x \in \mathbb{R}$ we get $E\Phi E_x\Phi^{-1}E^{-1} = e_xI$, where $e_x(y) := e^{ixy}$ for $y \in \mathbb{R}$. It was shown in [9, Lemma 3.8] (see also [2, Lemma 10.1] for $p = 2$) that $\operatorname*{s-lim}_{n\to\infty} e_{-\mu_n}\widetilde{K}e_{\mu_n}I = 0$ on $L^p(\mathbb{R})$ for every sequence μ_n tending to $+\infty$ or to $-\infty$. Therefore

$$\operatorname*{s-lim}_{n\to\infty}E_{\mu_n}^{-1}KE_{\mu_n} = \operatorname*{s-lim}_{n\to\infty}\Phi^{-1}E^{-1}e_{-\mu_n}\widetilde{K}e_{\mu_n}E\Phi = 0 \quad \text{on} \quad L^p(\mathbb{R}_+).$$

Analogously, $(E_{\mu_n}^{-1}KE_{\mu_n})^* = E_{\mu_n}^{-1}K^*E_{\mu_n}$ converges strongly to zero on the space $L^q(\mathbb{R}_+)$. Thus, for every sequence $\mu = \{\mu_n\}_{n=1}^\infty \subset \mathbb{R}$ converging to $s \in \{-\infty, +\infty\}$, we have $K_{\mathcal{E}^s_\mu} = 0$. ☐

Lemma 4.7. *Suppose $g \in SO(\mathbb{R}_+)$. Let $\mu = \{\mu_n\}_{n=1}^{\infty} \subset \mathbb{R}$ be a test sequence relative to $s \in \{-\infty, +\infty\}$. Then the limit operators $(gI)_{\mathcal{E}_\mu^s}$ and $S_{\mathcal{E}_\mu^s}$ with respect to the sequence of pseudoisometries $\mathcal{E}_\mu^s := \{E_{\mu_n}\}_{n=1}^{\infty} \subset \mathcal{B}$ exist and are given by*

$$(gI)_{\mathcal{E}_\mu^{\pm\infty}} = gI, \quad S_{\mathcal{E}_\mu^{\pm\infty}} = \pm I.$$

Proof. Let $s = +\infty$. It is easy to see that

$$E_{\mu_n}^{-1}(gI)E_{\mu_n} = gI, \quad (E_{\mu_n}^{-1}(gI)E_{\mu_n})^* = (gI)^*$$

for all $n \in \mathbb{N}$. Hence $(gI)_{\mathcal{E}_\mu^{+\infty}} = gI$.

Let us show that $S_{\mathcal{E}_\mu^{+\infty}} = I$. From Theorem 3.2 it follows that

$$E_{\mu_n}^{-1} S E_{\mu_n} = \Phi^{-1} \operatorname{Co}(s_{p,\mu_n})\Phi,$$

where $s_{p,\mu_n}(x) := s_p(x + \mu_n)$ for $x \in \mathbb{R}$. Hence we must show that

$$\lim_{n\to\infty} \| \operatorname{Co}(s_{p,\mu_n} - 1)\psi\|_{L^p(\mathbb{R}_+,d\mu)} = 0 \tag{4.19}$$

for $\psi \in L^p(\mathbb{R}_+, d\mu)$.

According to [15, Chap. III, Section 2.2], for every $f \in L^p(\mathbb{R})$ and every $\varphi \in L^1(\mathbb{R})$ with $\int_{\mathbb{R}} \varphi(x)dx = 1$, we have

$$\lim_{\varepsilon\to 0} \|f * \varphi_\varepsilon - f\|_{L^p(\mathbb{R})} = 0, \tag{4.20}$$

where $\varphi_\varepsilon(x) := \varepsilon^{-1}\varphi(x/\varepsilon)$ for $x \in \mathbb{R}$ and $\varepsilon > 0$. Choosing now rapidly decreasing functions φ in the Schwarz space $\mathcal{S}(\mathbb{R})$ whose Fourier transforms $F\varphi$ have compact supports in $\mathbb{R}$, we derive from (4.20) that the set Y of the functions $f \in L^2(\mathbb{R}) \cap L^p(\mathbb{R})$, for which Ff has compact support in $\mathbb{R}$, is dense in $L^p(\mathbb{R})$. Hence the set D of all functions $\psi \in L^2(\mathbb{R}_+, d\mu) \cap L^p(\mathbb{R}_+, d\mu)$, for which the Mellin transform $M\psi$ has compact support in $\mathbb{R}$, is dense in $L^p(\mathbb{R}_+, d\mu)$. Obviously, it is sufficient to prove (4.19) for all $\psi \in D$.

Fix $\psi \in D$. Since the support of $M\psi$ is compact, there exists a function $\chi \in C_0^\infty(\mathbb{R})$ with a compact support K such that

$$\operatorname{Co}(s_{p,\mu_n} - 1)\psi = M^{-1}\chi(s_{p,\mu_n} - 1)M\psi. \tag{4.21}$$

From Theorem 3.1 and (3.1) it follows that

$$\|M^{-1}\chi(s_{p,\mu_n} - 1)M\psi\|_{L^p(\mathbb{R}_+,d\mu)} \le c_p\|\psi\|_{L^p(\mathbb{R}_+,d\mu)}\|\chi(s_{p,\mu_n} - 1)\|_V, \tag{4.22}$$

where $c_p := \|S\|_{\mathcal{B}(L^p(\mathbb{R}))}$ and $\|\cdot\|_V := \|\cdot\|_{L^\infty(\mathbb{R})} + V(\cdot)$. It remains to show that

$$\|\chi(s_{p,\mu_n} - 1)\|_V = \|\chi(s_{p,\mu_n} - 1)\|_{L^\infty(\mathbb{R})} + V\big(\chi(s_{p,\mu_n} - 1)\big) \to 0 \tag{4.23}$$

as $n \to \infty$. We have

$$\|\chi(s_{p,\mu_n}-1)\|_{L^\infty(\mathbb{R})} \le \|\chi\|_{L^\infty(\mathbb{R})} \sup_{x\in K} |s_p(x+\mu_n)-1| \tag{4.24}$$

and

$$\begin{aligned} V\big(\chi(s_{p,\mu_n}-1)\big) &= \int_{\mathbb{R}} \left| \frac{d}{dx}\big[\chi(x)(s_p(x+\mu_n)-1)\big] \right| dx \\ &\le \int_{\mathbb{R}} |\chi(x)|\,|s_p'(x+\mu_n)|dx + \int_{\mathbb{R}} |\chi'(x)|\,|s_p(x+\mu_n)-1|dx \\ &\le C \sup_{x\in K} |s_p'(x+\mu_n)| + C' \sup_{x\in K} |s_p(x+\mu_n)-1|, \end{aligned} \tag{4.25}$$

where

$$C := \int_{\mathbb{R}} |\chi(x)|dx < \infty, \quad C' := \int_{\mathbb{R}} |\chi'(x)|dx < \infty.$$

Taking into account that $\mu_n \to +\infty$ and

$$s_p(x+\mu_n) = \coth[\pi(x+\mu_n+i/p)], \;\; s_p'(x+\mu_n) = -\frac{\pi}{\sinh^2[\pi(x+\mu_n+i/p)]},$$

we see that

$$\lim_{n\to\infty} \sup_{x\in K} |s_p(x+\mu_n)-1| = 0, \quad \lim_{n\to\infty} \sup_{x\in K} |s_p'(x+\mu_n)| = 0. \tag{4.26}$$

Combining (4.24)–(4.26), we arrive at (4.23). From (4.21)–(4.23) we obtain (4.19). Since $(E_{\mu_n}^{-1} S E_{\mu_n})^* = E_{\mu_n}^{-1} S E_{\mu_n}$, this completes the proof for $s = +\infty$.

The proof for $s = -\infty$ is analogous. □

5. Localization

5.1. Algebras $\mathcal{Z}$, $\mathcal{F}$, and Λ. Let us consider

$$\begin{aligned} \mathcal{Z} &:= \mathrm{alg}_{\mathcal{B}}\big\{I, S, cR, K : \; c \in SO(\mathbb{R}_+), \; K \in \mathcal{K}\big\}, \\ \mathcal{F} &:= \mathrm{alg}_{\mathcal{B}}\big\{aI, S, W_\alpha, W_\alpha^{-1} : a \in SO(\mathbb{R}_+)\big\}, \\ \Lambda &:= \big\{A \in \mathcal{B} : \; AC - CA \in \mathcal{K} \text{ for all } C \in \mathcal{Z}\big\}. \end{aligned}$$

It is easy to see that Λ is a closed unital subalgebra of $\mathcal{B}$.

Theorem 5.1 ([6, Theorem 6.8])**.** *We have $\mathcal{K} \subset \mathcal{Z} \subset \mathcal{F} \subset \Lambda$.*

Lemma 5.2. *An operator $A \in \Lambda$ is Fredholm if and only if the coset $A^\pi := A + \mathcal{K}$ is invertible in the quotient algebra $\Lambda^\pi := \Lambda/\mathcal{K}$.*

The proof is straightforward.

5.2. Fredholmness of operators in the algebra Λ. By [6, Theorem 6.11], the maximal ideal space $M(\mathcal{Z}^\pi)$ of the commutative Banach algebra $\mathcal{Z}^\pi := \mathcal{Z}/\mathcal{K}$ is homeomorphic to the set $\{-\infty,+\infty\} \cup (\Delta \times \mathbb{R})$. Let

$$\begin{aligned} \mathcal{I}^\pi_{\pm\infty} &:= \mathrm{id}_{\mathcal{Z}^\pi}\big\{P^\pi_\mp, (gR)^\pi \ : \ g \in SO(\mathbb{R}_+)\big\}, \\ \mathcal{I}^\pi_{\xi,x} &:= \big\{Z^\pi \in \mathcal{Z}^\pi : \ (Z^\pi)\widehat{\ }(\xi,x) = 0\big\} \quad \text{for} \ \ (\xi,x) \in \Delta \times \mathbb{R}, \end{aligned}$$

where $(Z^\pi)\widehat{\ }$ is the Gelfand transform of Z^π, which was explicitly given in [6, Section 6]. Further, let $\mathcal{J}^\pi_{\pm\infty}$ and $\mathcal{J}^\pi_{\xi,x}$ be the closed two-sided ideals of the Banach algebra Λ^π generated by the ideals $\mathcal{I}^\pi_{\pm\infty}$ and $\mathcal{I}^\pi_{\xi,x}$ of the algebra $\mathcal{Z}^\pi$, respectively, and put

$$\Lambda^\pi_{\pm\infty} := \Lambda^\pi/\mathcal{J}^\pi_{\pm\infty}, \quad \Lambda^\pi_{\xi,x} := \Lambda^\pi/\mathcal{J}^\pi_{\xi,x}$$

for the corresponding quotient algebras.

Theorem 5.3 ([6, Theorem 6.12])**.** *An operator $A \in \Lambda$ is Fredholm on the space $L^p(\mathbb{R}_+)$ if and only if the following two conditions are fulfilled:*

- (i) *the cosets $A^\pi + \mathcal{J}^\pi_{\pm\infty}$ are invertible in the quotient algebras $\Lambda^\pi_{\pm\infty}$, respectively;*
- (ii) *for every $(\xi,x) \in \Delta \times \mathbb{R}$, the coset $A^\pi + \mathcal{J}^\pi_{\xi,x}$ is invertible in the quotient algebra $\Lambda^\pi_{\xi,x}$.*

5.3. Quotient algebras $\Lambda_{+\infty}$ and $\Lambda_{-\infty}$. Let $\mathcal{J}_{\pm\infty}$ be the closed two-sided ideal of the algebra Λ generated by the operator $P_\mp$ and the ideal $\mathcal{K}$. By $\Lambda_{\pm\infty}$ denote the quotient algebra $\Lambda/\mathcal{J}_{\pm\infty}$.

It is not difficult to see that $R \in \mathrm{id}_{\mathcal{A}}\{P_-\} \cap \mathrm{id}_{\mathcal{A}}\{P_+\}$. Hence the ideals $\mathcal{J}^\pi_{\pm\infty}$ of the quotient algebra Λ^π can be also represented in the form $\mathcal{J}^\pi_{\pm\infty} = \mathrm{id}_{\Lambda^\pi}\{P^\pi_\mp\}$, and therefore, respectively,

$$\mathcal{J}^\pi_{\pm\infty} = \{A^\pi : \ A \in \mathcal{J}_{\pm\infty}\}. \tag{5.1}$$

Lemma 5.4. *Suppose $C_-, C_+ \in \Lambda$.*

- (a) *The invertibility of the coset $(C_+P_+ + C_-P_-)^\pi + \mathcal{J}^\pi_{+\infty}$ in the quotient algebra $\Lambda^\pi_{+\infty}$ is equivalent to the invertibility of the coset $C_+ + \mathcal{J}_{+\infty}$ in the quotient algebra $\Lambda_{+\infty}$.*
- (b) *The invertibility of the coset $(C_+P_+ + C_-P_-)^\pi + \mathcal{J}^\pi_{-\infty}$ in the quotient algebra $\Lambda^\pi_{-\infty}$ is equivalent to the invertibility of the coset $C_- + \mathcal{J}_{-\infty}$ in the quotient algebra $\Lambda_{-\infty}$.*

Proof. (a) Consider the mapping $\varphi : \Lambda/\mathcal{J}_{+\infty} \to \Lambda^\pi/\mathcal{J}^\pi_{+\infty}$ given by

$$A + \mathcal{J}_{+\infty} \mapsto A^\pi + \mathcal{J}^\pi_{+\infty} \quad (A \in \Lambda).$$

Obviously, φ is a homomorphism of $\Lambda/\mathcal{J}_{+\infty}$ onto $\Lambda^\pi/\mathcal{J}^\pi_{+\infty}$. If $A^\pi \in \mathcal{J}^\pi_{+\infty}$, then from (5.1) it follows that $A \in \mathcal{J}_{+\infty}$, and therefore φ is injective. So, φ is an isomorphism. Then $C_+ + \mathcal{J}_{+\infty} = \varphi^{-1}(C^\pi_+ + \mathcal{J}^\pi_{+\infty})$ is invertible in $\Lambda/\mathcal{J}_{+\infty}$ if and only if $C^\pi_+ + \mathcal{J}^\pi_{+\infty}$ is invertible in $\Lambda^\pi/\mathcal{J}^\pi_{+\infty}$. By the definition of the ideal $\mathcal{J}^\pi_{+\infty}$, we have $(C_\pm P_-)^\pi \in \mathcal{J}^\pi_{+\infty}$. From this observation and $P_+ + P_- = I$ it follows that

$$(C_+P_+ + C_-P_-)^\pi + \mathcal{J}^\pi_{+\infty} = C^\pi_+ + \mathcal{J}^\pi_{+\infty},$$

which finishes the proof of part (a). The proof of part (b) is analogous. $\square$

6. Invertibility of binomial functional operators

6.1. Invertibility of functional operators with slowly oscillating data. For $s \in \{0, \infty\}$, $a, b \in SO(\mathbb{R}_+)$, and $\alpha \in SOS(\mathbb{R}_+)$, put

$$L_*(s; a, b, \alpha) := \liminf_{t\to s} \Big(|a(t)| - |b(t)|\big(\alpha'(t)\big)^{-1/p}\Big),$$
$$L^*(s; a, b, \alpha) := \limsup_{t\to s} \Big(|a(t)| - |b(t)|\big(\alpha'(t)\big)^{-1/p}\Big).$$

Theorem 6.1 ([6, Theorem 1.1])**.** *Suppose $a, b \in SO(\mathbb{R}_+)$ and $\alpha \in SOS(\mathbb{R}_+)$. The functional operator $aI - bW_\alpha$ is invertible on the Lebesgue space $L^p(\mathbb{R}_+)$ if and only if either*

$$\inf_{t\in\mathbb{R}_+} |a(t)| > 0, \quad L_*(0; a, b, \alpha) > 0, \quad L_*(\infty; a, b, \alpha) > 0; \tag{6.1}$$

or

$$\inf_{t\in\mathbb{R}_+} |b(t)| > 0, \quad L^*(0; a, b, \alpha) < 0, \quad L^*(\infty; a, b, \alpha) < 0. \tag{6.2}$$

If (6.1) *holds, then*

$$(aI - bW_\alpha)^{-1} = \sum_{n=0}^{\infty} (a^{-1}bW_\alpha)^n a^{-1}I.$$

If (6.2) *holds, then*

$$(aI - bW_\alpha)^{-1} = -W_\alpha^{-1} \sum_{n=0}^{\infty} (b^{-1}aW_\alpha^{-1})^n b^{-1}I.$$

6.2. Invertibility of auxiliary binomial functional operators. Suppose $\alpha_0(t) := t$ and $\alpha_n(t) := \alpha[\alpha_{n-1}(t)]$ for $n \in \mathbb{Z}$ and $t \in \mathbb{R}_+$. Fix a point $\tau \in \mathbb{R}_+$ and put

$$\tau_- := \lim_{n\to-\infty} \alpha_n(\tau), \quad \tau_+ := \lim_{n\to+\infty} \alpha_n(\tau).$$

Then either $\tau_- = 0$ and $\tau_+ = \infty$, or $\tau_- = \infty$ and $\tau_+ = 0$. Let γ be a segment of $\mathbb{R}_+$ with endpoints τ and $\alpha(\tau)$. Suppose χ_γ is the characteristic function of γ and $\widetilde{\chi}_\gamma$ is an arbitrary function in $C(\mathbb{R}_+)$ with nonempty support in γ. Consider the half-open intervals

$$\gamma_- := \bigcup_{k=1}^{\infty} \alpha_{-k}(\gamma), \quad \gamma_+ := \bigcup_{k=1}^{\infty} \alpha_k(\gamma).$$

Let $\widetilde{\tau}_\pm$ denote the endpoint of the half-open interval $\gamma \cup \gamma_\pm$ such that $\widetilde{\tau}_\pm \neq \tau_\pm$, respectively. Consider functions $\chi_\pm \in C(\overline{\mathbb{R}}_+)$ such that $\chi_-(t) = 1$ for all $t \in \gamma_-$, $\chi_+(t) = 1$ for all $t \in \gamma_+$, and $\chi_-(t) + \chi_+(t) = 1$ for all $t \in \mathbb{R}_+$.

Lemma 6.2. *Let $A = aI - bW_\alpha$ where $a, b \in SO(\mathbb{R}_+)$ and $\alpha \in SOS(\mathbb{R}_+)$.*

(a) *Suppose*

$$L_*(\tau_-; a, b, \alpha) > 0 > L^*(\tau_+; a, b, \alpha), \tag{6.3}$$

$$\inf_{t\in\gamma\cup\gamma_-} |a(t)| > 0, \quad \inf_{t\in\gamma\cup\gamma_+} |b(t)| > 0, \tag{6.4}$$

and put

$$\widetilde{a}(t) := \begin{cases} a(t) & \text{for } t \in \gamma \cup \gamma_-, \\ a(\widetilde{\tau}_-) & \text{otherwise,} \end{cases} \quad \widetilde{b}(t) := \begin{cases} b(t) & \text{for } t \in \gamma \cup \gamma_+, \\ b(\widetilde{\tau}_+) & \text{otherwise.} \end{cases} \tag{6.5}$$

Then the operators

$$A_{1,\chi_-} := \widetilde{a}I - b\chi_- W_\alpha, \quad A_{2,\chi_+} := a\chi_+ I - \widetilde{b}W_\alpha \tag{6.6}$$

are invertible on the space $L^p(\mathbb{R}_+)$ and

$$A\Pi_r = 0, \quad \widetilde{\chi}_\gamma \Pi_r = \widetilde{\chi}_\gamma I \quad \text{for} \quad \Pi_r := (A_{1,\chi_-}^{-1} - A_{2,\chi_+}^{-1}) a\chi_\gamma I. \tag{6.7}$$

(b) *Suppose*

$$L^*(\tau_-; a, b, \alpha) < 0 < L_*(\tau_+; a, b, \alpha), \tag{6.8}$$

$$\inf_{t\in\gamma\cup\gamma_+} |a(t)| > 0, \quad \inf_{t\in\gamma\cup\gamma_-} |b(t)| > 0,$$

and put

$$\widetilde{a}(t) := \begin{cases} a(t) & \text{for } t \in \gamma \cup \gamma_+, \\ a(\widetilde{\tau}_+) & \text{otherwise,} \end{cases} \quad \widetilde{b}(t) := \begin{cases} b(t) & \text{for } t \in \gamma \cup \gamma_-, \\ b(\widetilde{\tau}_-) & \text{otherwise.} \end{cases}$$

Then the operators the operators

$$A_{1,\chi_+\circ\alpha} := \widetilde{a}I - b(\chi_+ \circ \alpha)W_\alpha, \quad A_{2,\chi_-} := a\chi_- I - \widetilde{b}W_\alpha,$$

are invertible on the space $L^p(\mathbb{R}_+)$ and

$$\Pi_l A = 0, \quad \Pi_l \widetilde{\chi}_\gamma I = \widetilde{\chi}_\gamma I \quad \text{for} \quad \Pi_l := \chi_\gamma a(A_{1,\chi_+\circ\alpha}^{-1} - A_{2,\chi_-}^{-1}).$$

Proof. (a) The idea of the proof is borrowed from [7, Lemma 3]. Clearly, the functions defined by (6.5) belong to $SO(\mathbb{R}_+)$. From (6.3)–(6.5) it follows that

$$\inf_{t\in\mathbb{R}_+}|\widetilde{a}(t)|>0,\quad \inf_{t\in\mathbb{R}_+}|\widetilde{b}(t)|>0,\quad L_*(\tau_\pm;\widetilde{a},b\chi_-,\alpha)>0,\quad L^*(\tau_\pm;a\chi_+,\widetilde{b},\alpha)<0.$$

By Theorem 6.1, the operators (6.6) are invertible on the space $L^p(\mathbb{R}_+)$, and

$$A^{-1}_{1,\chi_-}=\sum_{n=0}^{\infty}(\widetilde{a}^{-1}b\chi_-W_\alpha)^n\widetilde{a}^{-1}I,\ A^{-1}_{2,\chi_+}=-W_\alpha^{-1}\sum_{n=0}^{\infty}(\widetilde{b}^{-1}a\chi_+W_\alpha^{-1})^n\widetilde{b}^{-1}I. \tag{6.9}$$

Further, in view of (6.5), we get the relations

$$\begin{aligned}&(aI-bW_\alpha)W_\alpha^n\chi_\gamma I=(\widetilde{a}I-b\chi_-W_\alpha)W_\alpha^n\chi_\gamma I &&(n\in\mathbb{N}\cup\{0\}),\\ &(aI-bW_\alpha)(W_\alpha^{-1})^n\chi_\gamma I=(a\chi_+I-\widetilde{b}W_\alpha)(W_\alpha^{-1})^n\chi_\gamma I &&(n\in\mathbb{N}).\end{aligned} \tag{6.10}$$

Applying (6.9) and (6.10) we infer that

$$\begin{aligned}AA^{-1}_{1,\chi_-}a\chi_\gamma I=A_{1,\chi_-}A^{-1}_{1,\chi_-}a\chi_\gamma I=a\chi_\gamma I,\\ AA^{-1}_{2,\chi_+}a\chi_\gamma I=A_{2,\chi_+}A^{-1}_{2,\chi_+}a\chi_\gamma I=a\chi_\gamma I,\end{aligned}$$

whence $A\Pi_r=A(A^{-1}_{1,\chi_-}-A^{-1}_{2,\chi_+})a\chi_\gamma I=0$. On the other hand, since

$$\widetilde{\chi}_\gamma W_\alpha^n\chi_\gamma I=0,\quad \widetilde{\chi}_\gamma(W_\alpha^{-1})^n\chi_\gamma I=0\quad (n\in\mathbb{N}),$$

we deduce from (6.5) that

$$\begin{aligned}\widetilde{\chi}_\gamma\Pi_r&=\widetilde{\chi}_\gamma\sum_{n=0}^{\infty}(\widetilde{a}^{-1}b\chi_-W_\alpha)^n\widetilde{a}^{-1}a\chi_\gamma I+\widetilde{\chi}_\gamma W_\alpha^{-1}\sum_{n=0}^{\infty}(\widetilde{b}^{-1}a\chi_+W_\alpha^{-1})^n\widetilde{b}^{-1}a\chi_\gamma I\\ &=\widetilde{\chi}_\gamma\chi_\gamma I=\widetilde{\chi}_\gamma I,\end{aligned}$$

which completes the proof of (6.7). Part (a) is proved.

The proof of part (b) is similar and therefore is omitted. ☐

6.3. Invertibility of functional operators with multiplicative shifts.

Lemma 6.3. *Let $a,b\in SO(\mathbb{R}_+)$ and $\alpha:\mathbb{R}_+\to\mathbb{R}_+$ be a multiplicative shift given by $\alpha(t)=kt$ for all $t\in\mathbb{R}_+$ with some $k\in\mathbb{R}_+$. The following statements are equivalent:*

(i) *the functional operator $aI-bW_\alpha$ is invertible on the space $L^p(\mathbb{R}_+)$;*
(ii) *the coset $aI-bW_\alpha+\mathcal{J}_{+\infty}$ is invertible in the quotient algebra $\Lambda_{+\infty}$;*
(iii) *the coset $aI-bW_\alpha+\mathcal{J}_{-\infty}$ is invertible in the quotient algebra $\Lambda_{-\infty}$.*

Proof. (i) $\Rightarrow$ (ii). Clearly, if $A := aI - bW_\alpha$ is invertible on $L^p(\mathbb{R}_+)$, then it is Fredholm. Moreover, $A \in \Lambda$ by Theorem 5.1. Then from Lemma 5.2 we see that there exists $B \in \Lambda$ such that $AB - I \in \mathcal{K}$ and $BA - I \in \mathcal{K}$. Since the ideal $\mathcal{J}_{+\infty}$ contains $\mathcal{K}$, the latter relations imply that the coset $A + \mathcal{J}_{+\infty}$ is invertible in the quotient algebra $\Lambda_{+\infty}$ and $B + \mathcal{J}_{+\infty}$ is its inverse.

(ii) $\Rightarrow$ (i). Consider the test sequence $\nu := \{\nu_n\}_{n=1}^\infty \subset \mathbb{R}$ relative to $+\infty$ and given by

$$\nu_n := \begin{cases} 2\pi n |\log k|^{-1} & \text{if} \quad k \neq 1, \\ 2\pi n & \text{if} \quad k = 1 \end{cases} \qquad (n \in \mathbb{N}).$$

Then for every $n \in \mathbb{N}$, $f \in L^p(\mathbb{R}_+)$, and $t \in \mathbb{R}_+$,

$$(E_{\nu_n}^{-1} W_\alpha E_{\nu_n} f)(t) = t^{-i\nu_n} (kt)^{i\nu_n} f(kt) = k^{i\nu_n} f(kt) = (W_\alpha f)(t)$$

because $k^{i\nu_n} = 1$. That is, $E_{\nu_n}^{-1} W_\alpha E_{\nu_n} = W_\alpha$. Thus, the limit operator $(W_\alpha)_{\mathcal{E}_\nu^{+\infty}}$ with respect to the sequence of pseudoisometries $\mathcal{E}_\nu^{+\infty} := \{E_{\nu_n}\}_{n=1}^\infty$ exists and

$$(W_\alpha)_{\mathcal{E}_\nu^{+\infty}} = W_\alpha. \tag{6.11}$$

From Lemma 4.7 we obtain

$$(aI)_{\mathcal{E}_\nu^{+\infty}} = aI, \quad (bI)_{\mathcal{E}_\nu^{+\infty}} = bI, \quad S_{\mathcal{E}_\nu^{+\infty}} = I. \tag{6.12}$$

On the other hand, by Lemma 4.6,

$$K_{\mathcal{E}_\nu^{+\infty}} = 0 \quad \text{for every} \quad K \in \mathcal{K}. \tag{6.13}$$

Combining (6.11)–(6.13) with Lemma 4.1(b)–(c), we see that

$$(aI - bW_\alpha)_{\mathcal{E}_\nu^{+\infty}} = aI - bW_\alpha \tag{6.14}$$

and $J_{\mathcal{E}_\nu^{+\infty}} = 0$ for all $J \in \mathcal{J}_{+\infty}$ (recall that the ideal $\mathcal{J}_{+\infty}$ is generated by $\mathcal{K}$ and $P_- = (I - S)/2$, so $(P_-)_{\mathcal{E}_\nu^{+\infty}} = 0$).

Since the coset $aI - bW_\alpha + \mathcal{J}_{+\infty}$ is invertible in the quotient algebra $\Lambda_{+\infty}$, we deduce from Theorem 4.2 that the limit operator (6.14) is invertible. This finishes the proof of the implication (ii) $\Rightarrow$ (i). Thus, the equivalence (i) $\Leftrightarrow$ (ii) is proved.

The proof of the equivalence (i) $\Leftrightarrow$ (iii) is analogous. $\square$

7. Necessary conditions for Fredholmness

7.1. Necessity of condition (i). In this subsection we prove that condition (i) in Theorem 1.1 is necessary for the Fredholmness of the operator N. We start with the following auxiliary result.

Lemma 7.1. *Let the functions* χ_γ *and* $\widetilde{\chi}_\gamma$ *be as in Section* 6.2*, and* $n \in \mathbb{Z}$*.*

(a) *The operators* $(\chi_\gamma \circ \alpha_n)P_-P_+$ *and* $P_-P_+(\chi_\gamma \circ \alpha_n)I$ *are compact.*

(b) *The operators* $\widetilde{\chi}_\gamma P_\pm$ *and* $P_\pm \widetilde{\chi}_\gamma I$ *are not compact.*

Proof. (a) In view of (3.3), we have $P_-P_+ \in \mathrm{id}_{\mathcal{A}}\{R\}$. Let c be a continuous function equal 1 on the support of $\chi_\gamma \circ \alpha_n$ and vanishing at 0 and ∞. Then the operator $(c \circ \alpha_n)P_-P_+$ is compact by [6, Corollary 6.6]. Therefore the operator $(\chi_\gamma \circ \alpha_n)P_-P_+ = (\chi_\gamma \circ \alpha_n)(c \circ \alpha_n)P_-P_+$ is also compact. The compactness of $P_-P_+(\chi_\gamma \circ \alpha_n)I$ is proved analogously with the aid of Theorem 3.3. Part (a) is proved. Part (b) follows from [14, Theorem 4.1(c)]. $\square$

Lemma 7.2. *Suppose* $a, b, c, d \in SO(\mathbb{R}_+)$*,* $\alpha \in SOS(\mathbb{R}_+)$*, and the operator* N *is given by* (1.1)*.*

(a) *If* N *is Fredholm, then either*

$$L_*(0; a, b, \alpha) > 0, \quad L_*(\infty; a, b, \alpha) > 0; \tag{7.1}$$

or

$$L^*(0; a, b, \alpha) < 0, \quad L^*(\infty; a, b, \alpha) < 0. \tag{7.2}$$

(b) *If* N *is Fredholm, then either*

$$L_*(0; c, d, \alpha) > 0, \quad L_*(\infty; c, d, \alpha) > 0; \tag{7.3}$$

or

$$L^*(0; c, d, \alpha) < 0, \quad L^*(\infty; c, d, \alpha) < 0. \tag{7.4}$$

Proof. (a) Fix $s \in \{0, \infty\}$ and $\xi \in M_s(SO(\mathbb{R}_+))$. By Lemma 4.5, there exists a test sequence $h^\xi = \{h_n^\xi\}_{n=1}^\infty \subset \mathbb{R}_+$ relative to the point s such that the limit operator $N_{\mathcal{V}^s_{h^\xi}}$ with respect to the sequence of pseudoisometries $\mathcal{V}^s_{h^\xi} := \{V_{h_n^\xi}\}_{n=1}^\infty \subset \mathcal{B}$ exists and

$$N_{\mathcal{V}^s_{h^\xi}} = \big(a(\xi)I - b(\xi)W_{\alpha_\xi}\big)P_+ + \big(c(\xi)I - d(\xi)W_{\alpha_\xi}\big)P_-, \tag{7.5}$$

where $\alpha_\xi(t) = e^{\omega(\xi)}t$ is a multiplicative shift and $\omega(t) = \log[\alpha(t)/t]$ belongs to $SO(\mathbb{R}_+)$ in view of Lemma 2.5. From Lemma 4.4 it follows that $K_{\mathcal{V}^s_{h^\xi}} = 0$ for every $K \in \mathcal{K}$. Since the operator N is Fredholm, the coset $N^\pi = N + \mathcal{K}$ is invertible in the quotient algebra Λ^π in view of Lemma 5.2. Applying Theorem 4.2 with $\mathfrak{A} = \Lambda$, $\mathfrak{J} = \mathcal{K}$, $A = N$, and $\mathcal{U} = \mathcal{V}^s_{h^\xi}$, we conclude that the operator $N_{\mathcal{V}^s_{h^\xi}}$ is invertible.

Obviously, $N_{\mathcal{V}^s_{h^\xi}}$ is Fredholm. Then from Theorem 5.3 it follows that the coset $(N_{\mathcal{V}^s_{h^\xi}})^\pi + \mathcal{J}^\pi_{+\infty}$ is invertible in the quotient algebra $\Lambda^\pi_{+\infty}$. By Lemma 5.4, this is equivalent to the invertibility of the coset $a(\xi)I - b(\xi)W_{\alpha_\xi} + \mathcal{J}_{+\infty}$ in the quotient algebra $\Lambda_{+\infty}$. Since $\alpha_\xi(t) = e^{\omega(\xi)}t$ is a multiplicative shift, from

Lemma 6.3 it follows that the above condition is equivalent to the invertibility of the operator $a(\xi)I - b(\xi)W_{\alpha_\xi}$. Applying Theorem 6.1 to this operator and taking into account Lemma 2.8, we obtain either

$$|a(\xi)| - |b(\xi)|\big(\alpha'(\xi)\big)^{-1/p} = |a(\xi)| - |b(\xi)|\big(e^{\omega(\xi)}\big)^{-1/p} > 0 \tag{7.6}$$

or

$$|a(\xi)| - |b(\xi)|\big(\alpha'(\xi)\big)^{-1/p} = |a(\xi)| - |b(\xi)|\big(e^{\omega(\xi)}\big)^{-1/p} < 0. \tag{7.7}$$

The fibers $M_s(SO(\mathbb{R}_+))$ are connected compact Hausdorff spaces by Lemma 2.2. Since $a(\xi), b(\xi)$, and $\alpha'(\xi)$ depend continuously on $\xi \in M_s(SO(\mathbb{R}_+))$, we deduce that if N is Fredholm, then for every $s \in \{0,\infty\}$ either (7.6) holds for all $\xi \in M_s(SO(\mathbb{R}_+))$ or (7.7) holds for all $\xi \in M_s(SO(\mathbb{R}_+))$. Hence we conclude from Lemma 2.1 that for each $s \in \{0,\infty\}$ either $L_*(s;a,b,\alpha) > 0$ or $L^*(s;a,b,\alpha) < 0$.

It remains to prove that actually either (7.1) or (7.2) is fulfilled, that is, to show that $L^*(0;a,b,\alpha) < 0 < L_*(\infty;a,b,\alpha)$ or $L^*(\infty;a,b,\alpha) < 0 < L_*(0;a,b,\alpha)$ are impossible. Since either $\tau_- = 0$ and $\tau_+ = \infty$, or $\tau_- = \infty$ and $\tau_+ = 0$, the latter inequalities take either the form (6.3), or the form (6.8).

On the contrary, suppose (6.3) is fulfilled. Then there are open neighborhoods $u(\tau_\pm) \subset \mathbb{R}_+$ of $\tau_\pm$ such that $|a|$ is separated from zero on $u(\tau_-)$ and $|b|$ is separated from zero on $u(\tau_+)$. Take a segment $\gamma \subset \mathbb{R}_+ \setminus \overline{u(\tau_-) \cup u(\tau_+)}$ with endpoints τ and $\alpha(\tau)$. Since N is Fredholm, by a small perturbation of coefficients a, b in $C\big(\mathbb{R}_+ \setminus (u(\tau_-) \cup u(\tau_+))\big)$ we can achieve the fulfillment of (6.4) for perturbed coefficients keeping the operator N Fredholm. Notice that inequalities (6.3) remain valid for perturbed coefficients. Let us save notation a, b for perturbed coefficients. Then in virtue of Lemma 6.2(a) we obtain the operator Π_r given by (6.7). Setting now $A_+ := aI - bW_\alpha$ and $A_- := cI - dW_\alpha$ and taking into account Theorem 3.3, we get

$$NP_+ = (A_+P_+ + A_-P_-)P_+ \simeq P_+A_+ + (A_- - A_+)P_-P_+, \tag{7.8}$$

where $C \simeq D$ means that $C - D$ is a compact operator. Recall that since N is Fredholm, there is an operator $N^{(-1)} \in \mathcal{B}$, called a regularizer of N, such that $NN^{(-1)} \simeq N^{(-1)}N \simeq I$. Then, applying Π_r and $N^{(-1)}$, we infer from (7.8) that

$$P_+\Pi_r \simeq N^{(-1)}NP_+\Pi_r \simeq N^{(-1)}P_+A_+\Pi_r + N^{(-1)}(A_- - A_+)P_-P_+\Pi_r. \tag{7.9}$$

From Lemma 7.1(a) and

$$\Pi_r = \sum_{n=0}^{\infty}(\chi_\gamma \circ \alpha_n)(a^{-1}b\chi_-W_\alpha)^n + (\chi_\gamma \circ \alpha_{-n-1})W_\alpha^{-1}\sum_{n=0}^{\infty}(b^{-1}a\chi_+W_\alpha^{-1})^n b^{-1}aI$$

we get $P_-P_+\Pi_r \simeq 0$. On the other hand, by (6.7) with $A = A_+$, we obtain $A_+\Pi_r = 0$. The latter two relations imply in view of (7.9) that $P_+\Pi_r \simeq 0$. Let $\widetilde{\chi}_\gamma$ be as in Section 6.2. From (6.7) and Theorem 3.3 we get

$$P_+\widetilde{\chi}_\gamma I = P_+\widetilde{\chi}_\gamma\Pi_r \simeq \widetilde{\chi}_\gamma P_+\Pi_r \simeq 0.$$

Hence $P_+\widetilde{\chi}_\gamma I$ is a compact operator, which is impossible due to Lemma 7.1(b).

Analogously, if (6.8) holds, then applying Lemma 6.2(b) we conclude that

$$\Pi_l P_+ \simeq \Pi_l P_+ N N^{(-1)} \simeq \Pi_l (A_+ P_+ + P_- P_+ (A_- - A_+)) N^{(-1)} \simeq 0,$$

and hence

$$\widetilde{\chi}_\gamma P_+ = \Pi_l \widetilde{\chi}_\gamma P_+ \simeq \Pi_l P_+ \widetilde{\chi}_\gamma I \simeq 0,$$

which again is impossible. Thus, either (7.1) or (7.2) holds, and hence part (a) is proved. The proof of part (b) is analogous. □

Lemma 7.3. *Suppose $a, b, c, d \in SO(\mathbb{R}_+)$, $\alpha \in SOS(\mathbb{R}_+)$, and the operator N is given by* (1.1).

(a) *If N is Fredholm and* (7.1) *is fulfilled, then* $\inf\limits_{t\in\mathbb{R}_+} |a(t)| > 0$.

(b) *If N is Fredholm and* (7.2) *is fulfilled, then* $\inf\limits_{t\in\mathbb{R}_+} |b(t)| > 0$.

(c) *If N is Fredholm and* (7.3) *is fulfilled, then* $\inf\limits_{t\in\mathbb{R}_+} |c(t)| > 0$.

(d) *If N is Fredholm and* (7.4) *is fulfilled, then* $\inf\limits_{t\in\mathbb{R}_+} |d(t)| > 0$.

Proof. (a) Assume the contrary, that is, $\inf\limits_{t\in\mathbb{R}_+} |a(t)| = 0$. From (7.1) it follows that there exist numbers $0 < m < M < \infty$ such that the function a is bounded away from zero on $(0, m] \cup [M, \infty)$. Hence there is a point $t_0 \in (m, M)$ such that $a(t_0) = 0$. Fix some τ such that t_0 belongs to the interior of the segment γ with the endpoints τ and $\alpha(\tau)$. Choose m and M such that $\gamma \subset [m, M]$. Suppose $u = u(t_0)$ is a closed neighborhood of the point t_0 that is contained in γ and whose endpoints do not coincide with τ and $\alpha(\tau)$. Then there exists a continuous function χ_u supported in u and such that $\chi_u(t_0) = 1$.

Let $\varphi, \psi \in SO(\mathbb{R}_+)$ be functions such that

1. $a(t) = \varphi(t) = \psi(t)$ for $t \in (0, m] \cup [M, \infty)$;
2. $\varphi(t) = 0$ for $t \in u$ and $\varphi(t) \neq 0$ for $t \in \mathbb{R} \setminus u$;
3. $\psi(t) \neq 0$ for $t \in \mathbb{R}$;
4. $\varphi(t) = \psi(t)$ for $t \in \mathbb{R} \setminus \gamma$.

Consider the operator

$$\widetilde{N} = (\varphi I - bW_\alpha)P_+ + (cI - dW_\alpha)P_-.$$

It is clear that $\|\widetilde{N} - N\|_{\mathcal{B}} = O(\|\varphi - a\|_{L^\infty(\mathbb{R}_+)})$. Since φ can be chosen arbitrarily close to a in the norm of $L^\infty(\mathbb{R}_+)$ and N is Fredholm, we can guarantee that $\widetilde{N}$ is also Fredholm for some φ as above.

By Theorem 5.3, the coset $\widetilde{N}^\pi + \mathcal{J}^\pi_{+\infty}$ is invertible in the quotient algebra $\Lambda^\pi_{+\infty}$. Therefore, the coset $\varphi I - bW_\alpha + \mathcal{J}_{+\infty}$ is invertible in the quotient algebra $\Lambda_{+\infty}$ in view of Lemma 5.4(a). Then there exists an operator $B \in \Lambda$ such that

$$(\varphi I - bW_\alpha + \mathcal{J}_{+\infty})(B + \mathcal{J}_{+\infty}) = I + \mathcal{J}_{+\infty}. \tag{7.10}$$

On the other hand, $\inf_{t\in\mathbb{R}_+} |\psi(t)| > 0$, and by (7.1) we have for $s \in \{0, \infty\}$,

$$L_*(s; \psi, b, \alpha) = L_*(s; a, b, \alpha) > 0.$$

By Theorem 6.1, the operator $\psi I - bW_\alpha$ is invertible and

$$(\psi I - bW_\alpha)^{-1} = \sum_{n=0}^{\infty} \left(\frac{b}{\psi} W_\alpha\right)^n \frac{1}{\psi} I = \frac{1}{\psi} \sum_{n=0}^{\infty} \left(\frac{b}{\psi \circ \alpha} W_\alpha\right)^n.$$

Let

$$C := \chi_u \psi (\psi I - bW_\alpha)^{-1}.$$

From Theorem 5.1 we see that $C \in \mathcal{F} \subset \Lambda$. From the choice of φ and ψ it follows that $\chi_u \varphi = 0$ and $\chi_u(\varphi \circ \alpha_k) = \chi_u(\psi \circ \alpha_k)$ for all $k \in \mathbb{N}$. Therefore,

$$C(\varphi I - bW_\alpha) = \chi_u \left(\sum_{n=0}^{\infty} \frac{b}{\psi \circ \alpha} W_\alpha\right) (\varphi I - bW_\alpha) = \chi_u \varphi I = 0.$$

Hence

$$(C + \mathcal{J}_{+\infty})(\varphi I - bW_\alpha + \mathcal{J}_{+\infty}) = \mathcal{J}_{+\infty}. \tag{7.11}$$

Multiplying (7.11) from the right by $B + \mathcal{J}_{+\infty}$ and taking into account (7.10), we obtain $C + \mathcal{J}_{+\infty} = \mathcal{J}_{+\infty}$. Then $C \in \mathcal{J}_{+\infty}$.

It is clear that $\chi_u \circ \alpha_k = 0$ for $k \in \mathbb{N}$. Then

$$C \chi_u I = \chi_u \sum_{n=0}^{\infty} \left(\frac{b}{\psi \circ \alpha} W_\alpha\right)^n \chi_u I = \chi_u^2 I \in \mathcal{J}_{+\infty}.$$

From Lemmas 4.6, 4.7, and 4.1 it follows that for an arbitrary sequence of pseudoisometries $\mathcal{E}_\mu^{+\infty} = \{E_{\mu_n}\}_{n=1}^{\infty} \subset \mathcal{B}$, the limit operators for all operators $J \in \mathcal{J}_{+\infty}$ are equal to zero. In particular, then $(\chi_u^2 I)_{\mathcal{E}_\mu^{+\infty}} = 0$. On the other hand, since $\chi_u^2 \in SO(\mathbb{R}_+)$, from Lemma 4.7 it also follows that $(\chi_u^2 I)_{\mathcal{E}_\mu^{+\infty}} = \chi_u^2 I \neq 0$. This contradiction shows that $\inf_{t\in\mathbb{R}_+} |a(t)| > 0$. Part (a) is proved.

(b) If the operator N is Fredholm, then the operator

$$\begin{aligned} -W_\alpha^{-1}&[(aI - bW_\alpha)P_+ + (cI - dW_\alpha)P_-] \\ &= [(b \circ \beta)I - (a \circ \beta)W_\beta]P_+ + [(d \circ \beta)I - (c \circ \beta)W_\beta]P_- \end{aligned} \tag{7.12}$$

is also Fredholm. Recall that $\beta \in SOS(\mathbb{R}_+)$ by Lemma 2.7. Then from Lemma 2.6 we see that $a \circ \beta, b \circ \beta \in SO(\mathbb{R}_+)$. Since β preserves the orientation, has only two fixed points 0 and ∞, and $\log \alpha'$ is bounded, we obtain for $s \in \{0, \infty\}$,

$$\begin{aligned} L^*(s; a, b, \alpha) &= -\liminf_{t\to s} \big((\alpha' \circ \beta)(t)\big)^{-1/p} \Big(|(b \circ \beta)(t)| - |(a \circ \beta)(t)|(\beta'(t))^{-1/p}\Big) \\ &\geq -\sup_{t\in\mathbb{R}_+} \big((\alpha' \circ \beta)(t)\big)^{-1/p} L_*(s; b \circ \beta, a \circ \beta, \beta). \end{aligned}$$

Hence (7.2) implies that $L_*(s; b\circ\beta, a\circ\beta, \beta) > 0$ for $s \in \{0, \infty\}$. Applying part (a) to the operator (7.12), we obtain

$$0 < \inf_{t\in\mathbb{R}_+} |(b\circ\beta)(t)| = \inf_{t\in\mathbb{R}_+} |b(t)|.$$

Part (b) is proved. Parts (c) and (d) are proved by analogy with parts (a) and (b), respectively. $\square$

Combining Lemmas 7.2–7.3 and Theorem 6.1, we arrive at the following part of Theorem 1.2.

Theorem 7.4. *Suppose $a, b, c, d \in SO(\mathbb{R}_+)$, $\alpha \in SOS(\mathbb{R}_+)$, and the operator N is given by* (1.1). *If the operator N is Fredholm on the space $L^p(\mathbb{R}_+)$, then the functional operators $A_+ := aI - bW_\alpha$ and $A_- := cI - dW_\alpha$ are invertible on the space $L^p(\mathbb{R}_+)$.*

7.2. Necessity of condition (ii). To finish the proof of Theorem 1.2, it remains to prove the following.

Theorem 7.5. *Suppose $a, b, c, d \in SO(\mathbb{R}_+)$, $\alpha \in SOS(\mathbb{R}_+)$, the operator N is given by* (1.1)*, and for every $\xi \in \Delta$ the function n_ξ is defined by* (1.2). *If the operator N is Fredholm on the space $L^p(\mathbb{R}_+)$, then $n_\xi(x) \neq 0$ for every pair $(\xi, x) \in \Delta \times \mathbb{R}$.*

Proof. Fix $\xi \in \Delta$. In the proof of Lemma 7.2 it was shown that if N is Fredholm, then the operator $N_{\mathcal{V}^s_{h^\xi}}$ given by (7.5) with $\alpha_\xi(t) = e^{\omega(\xi)}t$ is invertible. On the other hand, taking into account Theorem 3.2 and Lemma 3.4, we see that $N_{\mathcal{V}^s_{h^\xi}} = \Phi^{-1}\operatorname{Co}(n_\xi)\Phi$, where $n_\xi \in SAP_p$ is given by (1.2). Hence, $\operatorname{Co}(n_\xi)$ is invertible on the space $L^p(\mathbb{R}_+, d\mu)$. Then from Theorem 3.5 we deduce that $\inf\limits_{x\in\mathbb{R}} |n_\xi(x)| > 0$. Since $s \in \{0, \infty\}$ and $\xi \in M_s(SO(\mathbb{R}_+))$ were chosen arbitrarily, we conclude that $n_\xi(x) \neq 0$ for all $(\xi, x) \in \Delta \times \mathbb{R}$. $\square$

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